PROCEEDINGS OF IAM, V.6, N.2, 2017, pp.225-232

THE SOLUTION OF THE FUZZY LIFTING PROBLEM ON FUZZY COVERING SPACES

Erdal Guner[1] , Sebahattin Balci[2]

[1] Department of Mathematics, Faculty of Sciences, Ankara University, Ankara,Turkey
[2] Department of Mathematics, Faculty of Sciences, Kyrgyzstan-Turkey Manas University, Bishkek,Kyrgyzstan
guner@science.ankara.edu.tr, balci@manas.edu.kg

Abstract. Let be a fuzzy path connected topological space and be the fuzzy sheaf of fundamental groups over fuzzy topological spaces. It is shown that it is a fuzzy covering space. Also, it is given "Fuzzy Lifting Theorem" for the fuzzy sheaf.

Keywords: Fuzzy Path Connected Topological Space, Fuzzy Sheaf, Fuzzy Covering Space, Fuzzy Lifting Theorem.

AMS Subject Classification: 54A40, 54A05.

1. Introduction

The concept of a fuzzy set was discovered by Zadeh [14] and one of its earliest branches, the theory of fuzzy topology, was developed by Chang [1] and others. Recently, Zheng [15] and Wuyts [13] introduced the concept of a fuzzy path. Using this concept, Salleh and Md Tap [12] constructed the fundamental group of a fuzzy topological space. Gumus and Yildiz [5] formed an algebraic fuzzy sheaf by means of the fuzzy topological group. Guner and Balci [6] gave some characterizations concerned with the fuzzy sheaf of the fundamental groups. The sheaves constructed over topological spaces, which are horizontally topological and vertically algebraic structures, are very interesting spaces. The fuzzy sheaf concept is introduced over fuzzy topological space.In this paper,the fuzzy sheaf of fundamental groups over fuzzy topological spaces is shown a fuzzy covering space. Furthermore, General Lifting Theorem and its some results are adapted to the fuzzy sheaves [8].

Definition 1.1. Let F be a fuzzy set in X. The set

$$SuppF = F_0 = \{x \in X : F(x) > 0\}$$

is called the support of F ([2] , [3], [11]).

Definition 1.2. Let F be a fuzzy set in a fuzzy topological space (X,τ). If for any two fuzzy points a_λ and b_μ in F, there is a fuzzy path from a_λ to b_μ contained in F, then F is said to be fuzzy path connected in (X,τ).

If $F = X$ in the above definition, we call (X,τ) a fuzzy path connected space ([9], [12])

Definition 1.3. Let $\widetilde{X}$ and X be fuzzy topological spaces and $p:\widetilde{X}\to X$ be a fuzzy continuous mapping. A fuzzy set $U\subset X$ is said to be evenly fuzzy covered by p if U is fuzzy connected and open fuzzy set, and each fuzzy component of $p^{-1}(U)$ is an open fuzzy set that is mapped fuzzy homeomorphically onto U by p.

A fuzzy covering mapping is a fuzzy continuous surjective mapping $p:\widetilde{X}\to X$ such that $\widetilde{X}$ is fuzzy path connected and locally fuzzy path connected, and every fuzzy point $a_\lambda\in X$ has an evenly fuzzy covered neighborhood.

If $p:\widetilde{X}\to X$ is a fuzzy covering mapping, we call $\widetilde{X}$ a *fuzzy covering space of* X.

Definition 1.4. Let $\widetilde{X}, X$ and B be fuzzy topological spaces, $p:\widetilde{X}\to X$ be a fuzzy covering mapping and $\varphi: B\to X$ be any fuzzy continuous mapping. If the mapping $\widetilde{\varphi}: B\to\widetilde{X}$ is fuzzy continuous such that $p\circ\widetilde{\varphi}=\varphi,$ then $\widetilde{\varphi}$ is called a fuzzy lifting of φ.

Definition 1.5. A bijective mapping f of fuzzy topological space (X,τ_1) onto fuzzy topological space (Y,τ_2) is called a fuzzy homeomorphism if it is fuzzy continuous and fuzzy open([7], [8], [10]).

Definition 1.6. A mapping f of fuzzy topological space (X,τ_1) into a fuzzy topological space (Y,τ_2) is called a fuzzy sheaf if it is locally fuzzy homeomorphism ([4], [5]).

Let X be a fuzzy path connected topological space and H_{a_λ} be the fundamental group of X based for any $a_\lambda\in X,$ that is $H_{a_\lambda}=\pi_1(X,a_\lambda)$ [12]. Let $X=(X,x_p)$ be a pointed fuzzy topological space for an arbitrary fuzzy fixed point $x_p\in X.$ Let us denote the disjoint union of all fundamental groups obtained for each $a_\lambda\in X$ by $H,$ i.e., $H=\bigvee_{a_\lambda\in X} H_{a_\lambda}.$ H is a set over X and the mapping $\psi: H\to X$ defined by

$$\psi(\sigma_{a_\lambda})=\psi([\alpha(A)]_{a_\lambda})=a_\lambda$$

for any $\sigma_{a_\lambda}=[\alpha(A)]_{a_\lambda}\in H_{a_\lambda}\subset H$ is onto.

Now, let $W\subset X$ be a open fuzzy set. Define a mapping $s: W\to H$ such that

$$s(a_\lambda)=[\gamma^{-1}(H)*\alpha(A)*\gamma(G)]_{a_\lambda}$$

for each $a_\lambda \in W,$ where $[\alpha(A)]_{x_p} \in H_{x_p}$ is any element and $[\gamma(G)]$ is an arbitrary fixed fuzzy homotopy class defines an isomorphism between H_{a_λ} and H_{x_p}. Then the change of s depends on only the change of $\sigma_{x_p} = [\alpha(A)]_{x_p}$. Furthermore, $\psi \circ s = 1_W$. Let us denote the totality of the mappings s defined on W by $\Gamma(W,H)$.

If B is a fuzzy base for $X,$ then $B^* = \{s(W) : W \in B, \ s \in \Gamma(W,H)\}$ is a fuzzy base for H. The mappings ψ and s are fuzzy continuous in this topology. Moreover ψ is a locally fuzzy topological mapping. Then (H,ψ) is a fuzzy sheaf over X. (H,ψ) or only H is called " the fuzzy sheaf of the fundamental groups" over X [6].

The group $H_{a_\lambda} = \pi_1(X, a_\lambda)$ is called the stalk of the fuzzy sheaf H for any $a_\lambda \in X$. For any open fuzzy set $W \subset X,$ an element s of $\Gamma(W,H)$ is called a fuzzy section of the fuzzy sheaf H over W. The set $\Gamma(W,H)$ is a group with the pointwise operation of multiplication. Thus, H is fuzzy sheaf of groups over X. That is, H is an algebraic fuzzy sheaf.

The fuzzy sheaf H satisfies the following properties:

1. Let $W \subset X$ be an open fuzzy set. Then, a fuzzy section over W can be extended to a global fuzzy section over X.
2. Any two stalks of H are isomorphic with each other.
3. Let $W_1, W_2 \subset X$ be any two open fuzzy sets, $s_1 \in \Gamma(W_1, H)$ and $s_2 \in \Gamma(W_2, H)$. If $s_1(x_0) = s_2(x_0)$ for any fuzzy point $x_0 \in W_1 \cap W_2,$ then $s_1 = s_2$ over the whole $W_1 \cap W_2$.
4. Let $W \subset X$ be an open fuzzy set and $s_1, s_2 \in \Gamma(W,H)$. If $s_1(x_0) = s_2(x_0)$ for any fuzzy point $x_0 \in W,$ then $s_1 = s_2$ over the whole W.

2. The Fuzzy Lifting Theorem on Fuzzy Covering Spaces

In this section, it is shown that the fuzzy sheaf H is a fuzzy covering space of X. Later, it is given " Fuzzy Lifting Theorem" for this fuzzy sheaf.We can state the following theorem.

Theorem 2.1. Let H be the fuzzy sheaf of fundamental groups over (X, x_p) and W be an open fuzzy set in X. Then

$$H_{x_p} \cong \Gamma(W,H).$$

Proof. Let $W \subset X$ be an open fuzzy set and $s \in \Gamma(W,H)$. Then there exists a unique element $\sigma_{x_p} = [\alpha(A)]_{x_p} \subset H_{x_p}$ such that

$$s(a_\lambda) = [\gamma^{-1}(H) * \alpha(A) * \gamma(G)]_{a_\lambda}$$

for every $a_\lambda \in W$. That is, to each element of H_{x_p}, there correspondence only one element in $\Gamma(W,H)$. Let us denote this correspondence by

$$\Phi : H_{x_p} \to \Gamma(W,H)$$

such that $\Phi(\sigma_{x_p}) = s$ for any $\sigma_{x_p} \in H_{x_p}$. Let $\sigma^1_{x_p} = [\alpha_1(A_1)]_{x_p}$, $\sigma^2_{x_p} = [\alpha_2(A_2)]_{x_p}$ and $\sigma^1_{x_p}$, $\sigma^2_{x_p}$ determine the fuzzy sections $s_1, s_2 \in \Gamma(W,H)$, respectively. Then

$$s_1(a_\lambda) = [\gamma^{-1}(H) * \alpha_1(A_1) * \gamma(G)]_{a_\lambda}$$

and

$$s_2(a_\lambda) = [\gamma^{-1}(H) * \alpha_2(A_2) * \gamma(G)]_{a_\lambda}$$

for every $a_\lambda \in W$. Then $s_1(a_\lambda) \neq s_2(a_\lambda)$, if $\sigma^1_{x_p} \neq \sigma^2_{x_p}$. So Φ is one to one. As a result of definition of Φ, Φ is onto. Thus Φ is a bijection.

Φ is a homomorphism. Because, if $\sigma^1_{x_p} = [\alpha_1(A_1)]_{x_p}$, $\sigma^2_{x_p} = [\alpha_2(A_2)]_{x_p}$, then

$$\begin{aligned}\sigma^1_{x_p} \cdot \sigma^2_{x_p} &= [\alpha_1(A_1)]_{x_p} \cdot \alpha_2(A_2)]_{x_p} \\ &= [\alpha_1(A_1) * \alpha_2(A_2)]_{x_p}.\end{aligned}$$

So, the element $\sigma^1_{x_p} \cdot \sigma^2_{x_p} \in H_{x_p}$ defines a fuzzy section $s \in \Gamma(W,H)$ such that

$$s(a_\lambda) = [\gamma^{-1}(H) * (\alpha_1(A_1) * \alpha_2(A_2)) * \gamma(G)]_{a_\lambda}$$

for every $a_\lambda \in W$. On the other hand, for every $a_\lambda \in W$,

$$\begin{aligned}s_1(a_\lambda) \cdot s_1(a_\lambda) &= [\gamma^{-1}(H) * \alpha_1(A_1) * \gamma(G)]_{a_\lambda} \cdot \gamma^{-1}(H) * \alpha_2(A_2) * \gamma(G)]_{a_\lambda} \\ &= [\gamma^{-1}(H) * (\alpha_1(A_1) * \alpha_2(A_2)) * \gamma(G)]_{a_\lambda}.\end{aligned}$$

Thus,

$$\Phi(\sigma^1_{x_p} \cdot \sigma^2_{x_p}) = s = s_1 \cdot s_2 = \Phi(\sigma^1_{x_p}) \cdot \Phi(\sigma^2_{x_p}).$$

Therefore, Φ is an isomorphism.

We can state as a result of Theorem 2.1 that the stalk H_{x_p} completely determines the group of fuzzy sections over W. In particular, if we take $W = X$, then the stalk H_{x_p} completely determines the group of global fuzzy sections over X.

Now we can state the following corollary.

Corollary 2.1. Let H be the fuzzy sheaf of fundamental groups over fuzzy topological space X. Let H_{a_λ} be the stalk over the fuzzy point $a_\lambda \in X$ and W be an open fuzzy set. Then, $H_{a_\lambda} \cong \Gamma(W,H)$. Particularly, $H_{a_\lambda} \cong \Gamma(X,H)$.

According to this corollary, we can say that, if $\sigma_{a_\lambda} \in H_{a_\lambda}$ is any element and W is an open fuzzy set in X, then there is a unique fuzzy section $s \in \Gamma(W,H)$ such that $s(a_\lambda) = \sigma_{a_\lambda}$. Since

$$\psi|_{s(W)}: s(W) \to W$$

is a fuzzy topological mapping and $s = \left(\psi|_{s(W)}\right)^{-1}$,

$$\psi^{-1}(W) = \bigvee_{i \in I} s_i(W),\ s_i \in \Gamma(W,H)$$

and

$$\psi|_{s_i(W)}: s_i(W) \to W$$

is a fuzzy topological mapping. So, the open fuzzy set W is evenly fuzzy covered by ψ. Thus ψ is a fuzzy covering mapping and (H,ψ) is a fuzzy covering space of X.

Now, let $b_\mu \in X$ be any fuzzy point and $\gamma(C)$ be a fuzzy path initial point b_μ. Then the mapping

$$s \circ \gamma : I \to H$$

is a fuzzy continuous mapping and $\psi \circ (s \circ \gamma) = \gamma$. If we write $(s \circ \gamma)(b_\mu) = \rho_{b_\mu} \in H_{b_\mu}$, then $s \circ \gamma$ is a fuzzy lifting of fuzzy path $\gamma(C)$ from the initial point ρ_{b_μ} over b_μ in H. Write $s \circ \gamma(C) = (s \circ \gamma)(C) = \gamma^*(C)$, then $\gamma^*(C)$ is unique, because the mapping $\psi|_{s(X)}: s(X) \to X$ is a fuzzy homeomorphism.

We can then state the following theorem.

Theorem 2.2. Let (H,ψ) be the fuzzy sheaf of fundamental groups over fuzzy topological space X, $b_\mu \in X$ be any fuzzy point and $\gamma(C)$ be a fuzzy path with initial point b_μ in X. Then $\gamma(C)$ has a unique lifting $\gamma^*(C)$ with initial point ρ_{b_μ} in H_{b_μ}, for $\rho_{b_\mu} \in H_{b_\mu}$.

Now, we give the following theorem.

Theorem 2.3. Let (H,ψ) be the fuzzy sheaf of fundamental groups over fuzzy topological space X and suppose that $\gamma_1^*(C_1)$ and $\gamma_2^*(C_2)$ are fuzzy paths with common initial point ρ_{b_μ} and terminal point ρ_{c_η} in H. Then $\gamma_1^*(C_1)$ and $\gamma_2^*(C_2)$ are fuzzy homotopic paths in H if and only if $\psi \circ \gamma_1^*(C_1) = (\psi \circ \gamma_1^*)(C_1)$ and $\psi \circ \gamma_2^*(C_2) = (\psi \circ \gamma_2^*)(C_2)$ are fuzzy homotopic paths in X.

Proof. If $\gamma_1^*(C_1)$ is fuzzy homotopic to $\gamma_2^*(C_2)$ by a fuzzy homotopy G, then $\psi \circ G$ is a fuzzy homotopy between $(\psi \circ \gamma_1^*)(C_1)$ and $(\psi \circ \gamma_2^*)(C_2)$. For a proof of the other half of theorem, let b_μ and c_η denote the common initial point and common terminal point of $(\psi \circ \gamma_1^*)(C_1)$ and $(\psi \circ \gamma_2^*)(C_2)$, respectively. Let

$$H : (I, \tilde{\varepsilon}_I) \times (J, \tilde{\varepsilon}_J) \to (X, \tau)$$

be a fuzzy homotopy between $(\psi \circ \gamma_1^*)(C_1)$ and $(\psi \circ \gamma_2^*)(C_2)$. On the other hand, if $\rho_{b_\mu} \in H_{b_\mu}$, then there is a unique fuzzy section $s \in \Gamma(X,H)$ such that $s(b_\mu) = \rho_{b_\mu}$. So,

$$\left(s \circ (\psi \circ \gamma_1^*)\right)(C_1) = \gamma_1^*(C_1)$$

and

$$\left(s \circ (\psi \circ \gamma_2^*)\right)(C_2) = \gamma_2^*(C_2).$$

Furthermore, $s \circ H$ is a fuzzy homotopy between $\gamma_1^*(C_1)$ and $\gamma_2^*(C_2)$.

Thus, (H,ψ) is a regular fuzzy covering space of X.

Now, we can give " Fuzzy Lifting Theorem" for the fuzzy sheaf H.

Theorem 2.4. Let $X = (X, b_\mu)$, $Y = (Y, c_{\mu_1})$ be fuzzy path connected topological spaces, (H,ψ) be the fuzzy sheaf of fundamental groups over fuzzy pointed topological space (X, b_μ) and $\rho_{b_\mu} \in \psi^{-1}(b_\mu)$ be any fuzzy point. If

$$f : (Y, c_{\mu_1}) \to (X, b_\mu)$$

is any fuzzy continuous mapping, then f can be lifted to a unique fuzzy continuous

$$f^* : (Y, c_{\mu_1}) \to (H, \rho_{b_\mu})$$

such that $\psi \circ f = f^*$.

Proof. Let $f:(Y,c_{\mu_1})\to(X,b_\mu)$ be a fuzzy continuous mapping. Then $f(c_{\mu_1})=\rho_{b_\mu}$. If $\rho_{b_\mu}\in\psi^{-1}(b_\mu)$ is any fuzzy point, then there exists a unique fuzzy section $s\in\Gamma(X,H)$ such that $s(b_\mu)=\rho_{b_\mu}$. Thus,

$$s\circ f:(Y,c_{\mu_1})\to(H,\rho_{b_\mu})$$

is a fuzzy continuous mapping and

$$\psi\circ(s\circ f)=f.$$

So, $s\circ f$ is a fuzzy lifting of f to H. Let us denote $s\circ f$ by f^*. Since the fuzzy section s is unique, f^* is unique.

Finally, we can state the following theorem.

Theorem 2.5. Let $X=(X,b_\mu)$, $Y=(Y,c_{\mu_1})$ be fuzzy path connected topological spaces, (H,ψ) be the fuzzy sheaf of fundamental groups over fuzzy pointed topological space (X,b_μ) and $\rho_{b_\mu}\in\psi^{-1}(b_\mu)$ be any fuzzy point and

$$f^*,g^*:(Y,c_{\mu_1})\to(H,\rho_{b_\mu})$$

be any two fuzzy continuous mappings such that $\psi\circ f^*=\psi\circ g^*$. Then we have

$$f^*=g^*.$$

Proof. This is a result of Theorem 2.4.

3. Conclusion

In this paper, the fuzzy sheaf of fundamental groups is constructed over fuzzy path connected topological spaces . It is shown that this sheaf is a fuzzy covering space . Also, it is given “ Fuzzy Lifting Theorem” for this sheaf .

References

1. Chang C.L. Fuzzy topological spaces, J. Math. Anal. Appl.,V.24, 1968, pp.182-190.
2. Chon I. Some properties of fuzzy topological groups, Fuzzy Sets and Systems, V.123, 2001, pp.197-201.
3. Ferraro M., Foster D.H. C^1 fuzzy manifolds, Fuzzy Sets and Systems, V.54, 1993, pp.99-106.
4. Ganster M.,Georgiou D.N., Jafari S., On fuzzy topological groups and fuzzy continuous function, Hacet. J. Math. Stat., V.34, 2005, pp.35-43.
5. Gumus S., Yildiz C. On the fuzzy sheaf of the fuzzy groups formed by fuzzy topological spaces, Int.J.Contemp. Math. Sci.,V.3, N.5-8, 2008, pp.371-381.

6. Guner E., Balci S. On the fuzzy sheaf of the fundamental groups, Taiwanese J. Math., V.11, N.4, 2007, pp.1057-1062.
7. Johnson T.P. The group of L-fuzzy homeomorphisms and the group of permutations of the ground set, J.Math Anal. Appl., V.245, N.2, 2000, pp.423-429.
8. Johnson T.P. The group of fuzzy homeomorphisms, Fuzzy Sets and Systems,V.45, N.3, 1992, pp.355-358.
9. Klawonn F. Fuzzy unit interval and fuzzy paths , Applications of Category Theory to Fuzzy Subsets, 1989, pp.245-256.
10. Kumar S., Alam M.Z. A necesarry and sufficient condition for fuzzy homeomorphism , Acta Cienc. Indica Math.,V.19, N.3, 1993, pp.135-138.
11. Pu P.M., LiuY.M. Fuzzy topology I neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., V.76, 1980, pp.571-599.
12. Salleh A.R., Md A.O. Tap, The fundamental group of fuzzy topological spaces, Sains Malaysiana, V.15, N.4, 1986, pp.397-407.
13. Wuyts P., Fuzzy path and fuzzy connectedness, Fuzzy Sets and Systems V.24, N.1, 1987, pp.127-128.
14. Zadeh L.A., Fuzzy sets, İnformation and Control, 8, 1965, pp.338-365.
15. Zheng C.Y. Fuzzy path and fuzzy connectedness, Fuzzy Sets and Systems, V.14, 1984, 273-280.

Решение нечеткой поднимающейся проблемы на нечетких закрывающих пространствах

Эрдал Гюнер[1], Себахаттин Балчи[2]

[1]Математический факультет, факультет наук, Университет Анкары, Анкара, Турция
[2]Математический факультет, факультет наук, Кыргызстан-Турция Университет Манас, Бишкек, Кыргызстан
guner@science.ankara.edu.tr , balci@manas.edu.kg

РЕЗЮМЕ

Пусть имеется нечеткий пучок, связанный с топологическим пространством и являющийся нечетким пучком фундаментальных групп над нечеткими топологическими пространствами. Показано, что это нечеткое накрывающее пространство. Кроме того, для нечеткого пучка приводится «Теорема нечеткого поднятия».

Ключевые слова: Нечеткий путь связанный с топологическим пространством, нечеткий пучок, нечеткое пространство покрытия, нечеткая теорема о подъеме.